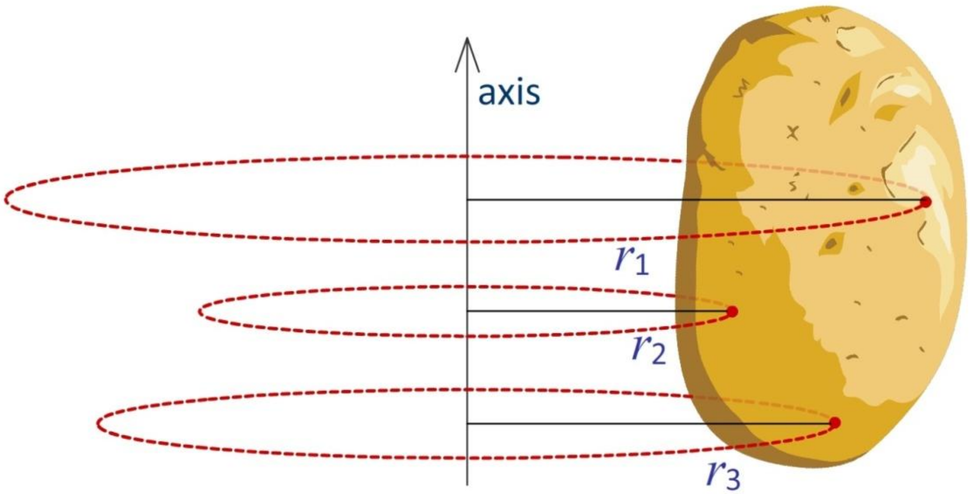
Rotational kinematics



V Vijaykiran
M.Sc., B.Ed., NET
SIGMA Physics Resource Centre
http://www.sigmaprc.in

Axis of rotation

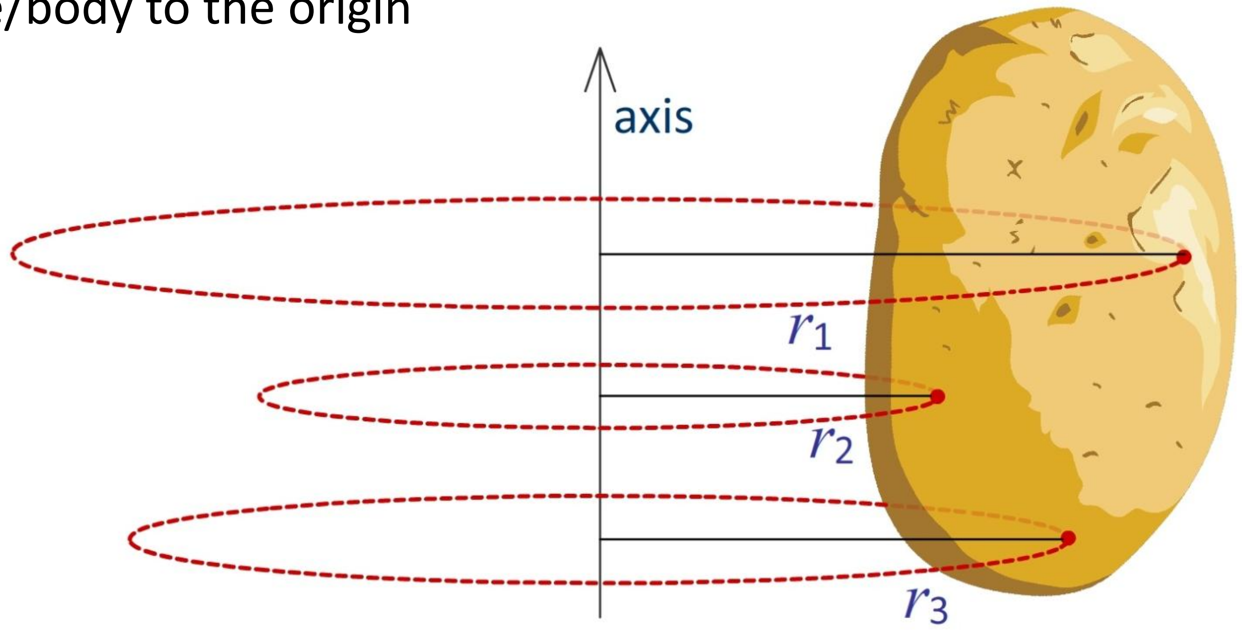
An imaginary line joining centers of all circular paths followed by particles of a body.

Radius vector

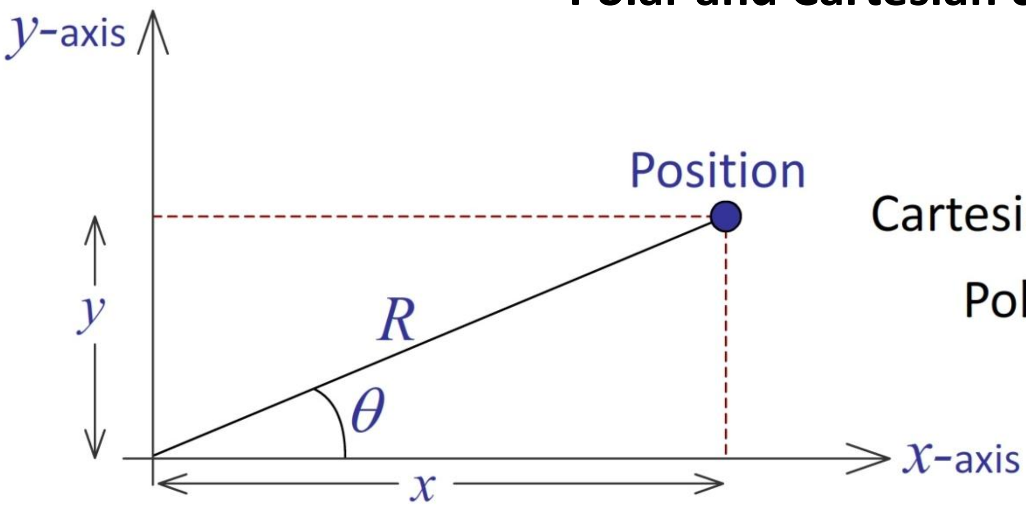
A vector drawn from the axis of rotation to the particle (perpendicular to the axis of rotation)

Position vector

A vector joining the particle/body to the origin



Polar and Cartesian coordinates



Cartesian coordinates (x, y)

Polar coordinates (R, θ)

Polar coordinates to Cartesian coordinates

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

Cartesian coordinates to Polar coordinates

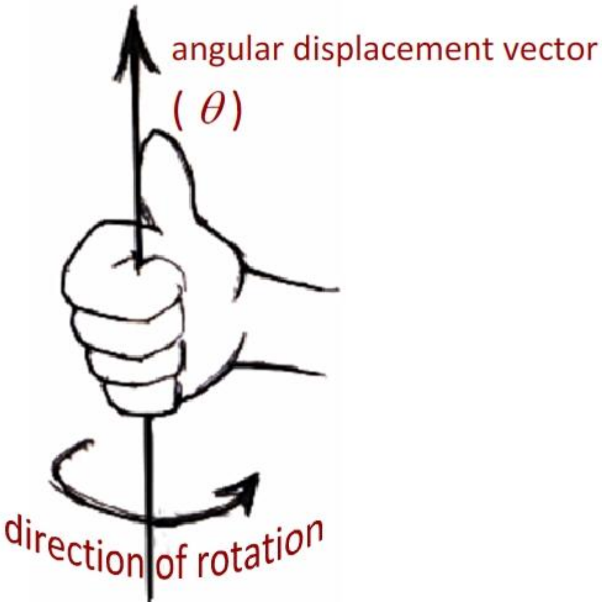
$$R = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar coordinates offer an advantage in analyzing circular motion.

Angular displacement (θ)

Displacement of a body in terms of angle through which the radius vector is rotated.

- Angular displacement is a vector quantity
- Direction of angular displacement is obtained using right hand thumb rule
- SI unit : radian (rad)
- Other units : degrees (deg), rotation (rot), revolution (rev)



Useful conversions

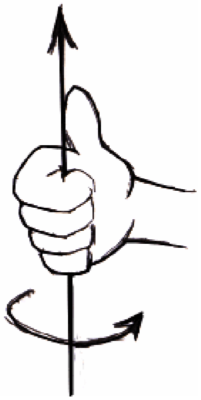
$$1 \text{ rotation or } 1 \text{ revolution} = 360^{\circ} = 2\pi \text{ rad}$$

$$1 \text{rad} = \frac{180}{\pi} \text{ degrees} \quad \Rightarrow \quad 1^{\circ} = \frac{\pi}{180} \text{ rad}$$

Angular velocity (ω)

Rate of change of angular displacement as a function of time.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

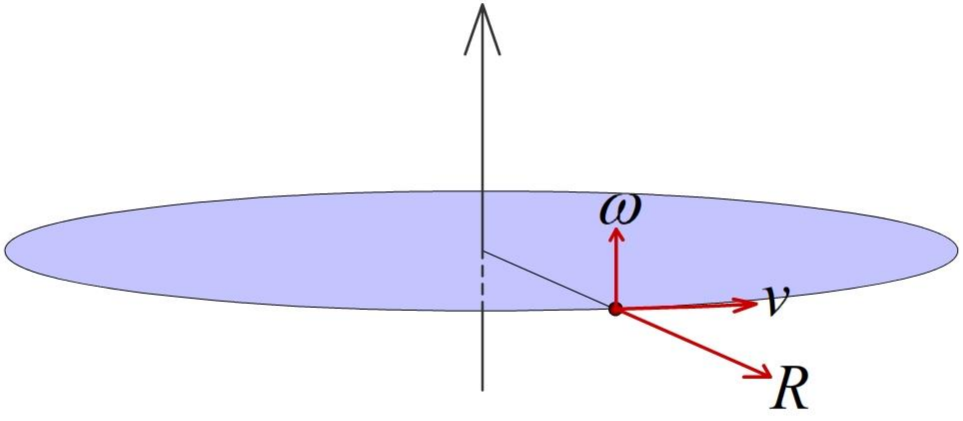


- Angular velocity is a vector quantity
- Direction of angular velocity is obtained using right hand thumb rule
- SI unit : rad s^{-1}
- Other units : rotation per minute (rpm) etc.

Useful conversions

$$1\text{rpm} = 6^{\circ} \text{ per second}$$

$$1\text{rpm} = \frac{\pi}{30}\ \text{rad}s^{-1}$$



Angular acceleration (α)

Angular acceleration is defined as the rate of change of angular velocity as a function of time.

$$\boxed{\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}}$$

since $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \alpha = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$

- Angular velocity is a vector quantity
- SI unit : rad s^{-2}

Equations of motion (for constant α)

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 - \omega_i^2 = 2\alpha\theta$$

$$\theta_n = \omega_i + \alpha\left(n - \frac{1}{2}\right)$$

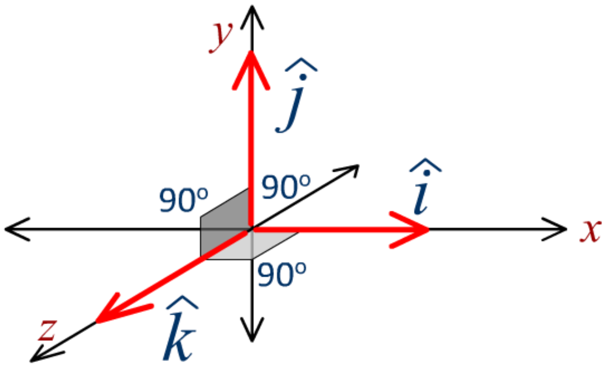
$$\omega_{avg} = \frac{\omega_i + \omega_f}{2}$$

Notice the similarity in the equations. Many such similarities are observed in the analysis of rotatory motion.

Vector product or cross product of vectors

Magnitude of cross product of two vectors is defined as the product of magnitudes of the two vectors and sine of angle between the vectors.

$$\boxed{\boldsymbol{A}\times\boldsymbol{B} = |A||B|\sin(\theta)\,\hat{n}}$$



If the vectors are given in their component forms i.e.

$$\boldsymbol{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\boldsymbol{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

then the cross product is given by

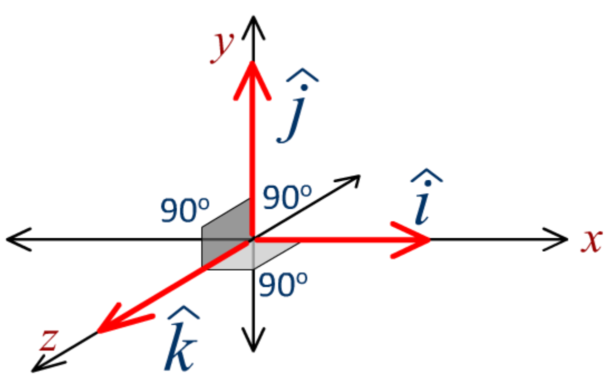
$$\boxed{\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}}$$

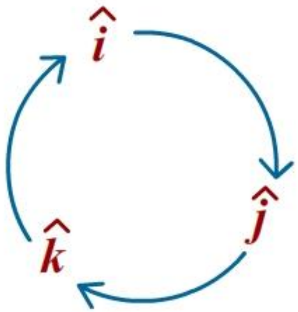
$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Direction is given by the right hand thumb rule

Properties of vector product (or cross product)

- Cross product of two parallel vectors is a null vector
- Magnitude of cross product of two mutually perpendicular vectors is equal to the product of their magnitudes
- Cross product is not commutative in nature $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$
- Cross product is distributive $\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$
- $\hat{i} \times \hat{j} = \hat{k}; \quad \hat{j} \times \hat{k} = \hat{i}\ ; \quad \hat{k} \times \hat{i} = \hat{j}$
- $\hat{j} \times \hat{i} = -\hat{k}; \quad \hat{k} \times \hat{j} = -\hat{i}\ ; \quad \hat{i} \times \hat{k} = -\hat{j}$
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \bar{O}$



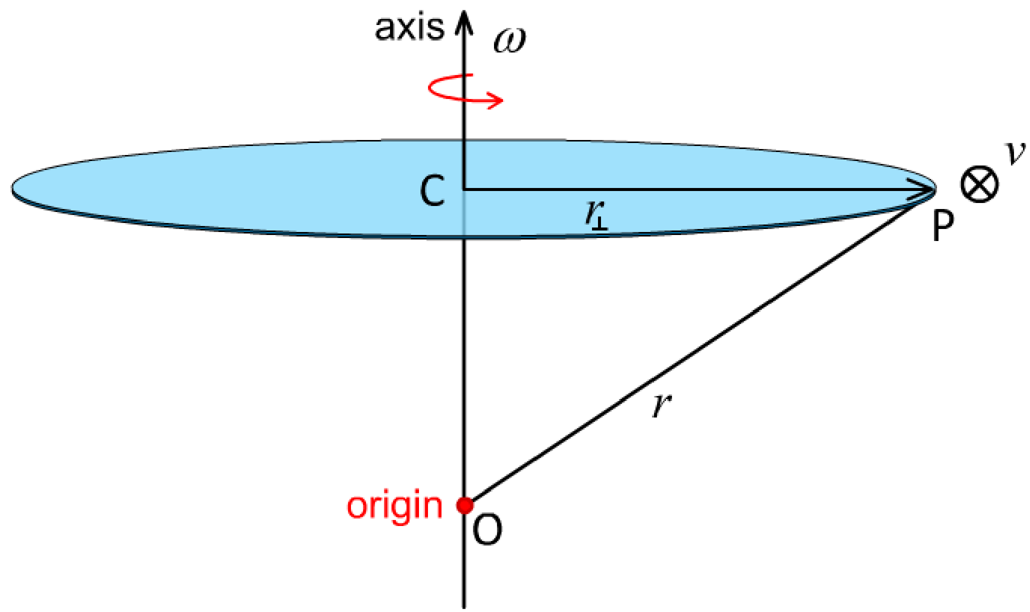


Along the arrows : +ve
Opposite to arrows : -ve

Relation between *v*, *R* and ω

Consider a disc rotating in the *x*-*y* plane (i.e. *z*-axis is the axis of rotation) with uniform angular velocity ω

Consider a point P on the disc at the instance shown in the figure. Let the distance of the point from an origin (O) be $\boldsymbol{r}$.



$$\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times \mathbf{OP}$$

$$\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times (\mathbf{OC} + \mathbf{CP})$$

$$\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times (\mathbf{OC}) + \boldsymbol{\omega} \times (\mathbf{CP})$$

$$\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times (\mathbf{CP})$$

$$\left|\boldsymbol{\omega} \times \boldsymbol{r}\right| = \omega \times r_{\square}$$

Linear velocity of a particle at that point is also of the same magnitude and tangential in direction. Therefore

$$\boxed{\bar{\boldsymbol{v}} = \bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{r}}}$$